

LARGE DEVIATIONS FOR PROCESSES ON HALF-LINE

F.C. KLEBANER, A.V. LOGACHOV, AND A.A. MOGULSKII

ABSTRACT. We consider a sequence of processes $X_n(t)$ defined on half-line $0 \leq t < \infty$. We give sufficient conditions for Large Deviation Principle (LDP) to hold in the space of continuous functions with metric $\rho_\kappa(f, g) = \sup_{t \geq 0} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}}$, $\kappa \geq 0$. LDP is established for Random Walks, Diffusions, and CEV model of ruin, all defined on the half-line. LDP in this space is “more precise” than that with the usual metric of uniform convergence on compacts.

Keywords: Large Deviations, Random Walk, Diffusion processes, CEV model.

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1. INTRODUCTION

In this work we derive sufficient conditions for a sequence $\{X_n\}_{n=1}^\infty$ of stochastic processes $X_n(t)$; $0 \leq t < \infty$, to satisfy the Large Deviation Principle (LDP) in the space of continuous functions on $[0, \infty)$, which we denote by \mathbb{C} .

In the recent literature [19], [9], [12] the space \mathbb{C} is considered with the metric

$$\rho^{(P)}(f, g) := \sum_{k=1}^{\infty} 2^{-k} \min\left\{ \sup_{0 \leq t \leq k} |f(t) - g(t)|, 1 \right\}. \quad (1.1)$$

[20] (Theorem 2.6) gives sufficient conditions for X_n to satisfy LDP in the space $(\mathbb{C}, \rho^{(P)})$.

As noted in [20], convergence $f_n \rightarrow f$ in metric $\rho^{(P)}$ is equivalent to convergence in $\mathbb{C}[0, T]$ with uniform metric for any $T \geq 0$. A considerable drawback of metric $\rho^{(P)}$ is that it is “not sensitive” to behaviour of functions as $t \rightarrow \infty$.

We consider the space \mathbb{C} with metric

$$\rho(f, g) = \rho_\kappa(f, g) := \sup_{t \geq 0} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}},$$

for a fixed $\kappa \geq 0$. It is obvious that (\mathbb{C}, ρ) is a complete separable metric (Polish) space.

As we shall see in Section 2, the LDP in the space (\mathbb{C}, ρ) is “more precise” than the LDP in $(\mathbb{C}, \rho^{(P)})$.

In this work we treat continuous processes on infinite interval. As we envisage, a treatment of discontinuous processes on infinite interval will need essentially different to ρ metric, (see [11], for the LDP for Compound Poisson processes on infinite

interval). Note that in [10], Theorem 1.3.27, LDP for Wiener process in space $(\mathbb{C}, \rho_\kappa)$ when $\kappa = 0$ is given, while in [7] the Law of Iterated Logarithm is proved for Wiener process in this space.

The paper is organised as follows. Sufficient conditions for LDP in the space (\mathbb{C}, ρ) are given in Section 2, Theorem 2.1. We also compare Theorem 2.1 and Theorem 2.6 of [20], and show that Theorem 2.1 is more precise. Next we apply Theorem 2.1 to different kind of processes, such as Random Walks and Diffusions on half line. Only Random Walks case is given here, and the reader referred to the Arxiv version for other examples.

2. MAIN RESULT

To formulate the main result we give a number of definitions. For any $T \in (0, \infty)$ denote by $\mathbb{C}[0, T]$ the metric space of real continuous functions $f = f(t)$; $0 \leq t \leq T$, with metric

$$\rho_T(f, g) := \sup_{0 \leq t \leq T} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}},$$

where $\kappa \geq 0$ is fixed.

We say that in space $\mathbb{C}[0, T]$ there is a (good) rate function

$$I_0^T = I_0^T(f) : \mathbb{C}[0, T] \rightarrow [0, \infty],$$

if: (i) it is lower semi-continuous: *for any* $f \in \mathbb{C}[0, T]$

$$\liminf_{f_n \rightarrow f} I_0^T(f_n) \geq I_0^T(f); \quad (2.1)$$

(ii) *for any* $r \geq 0$ the set

$$B_{T,r} := \{f \in \mathbb{C}[0, T] : I_0^T(f) \leq r\}$$

is a compact in $\mathbb{C}[0, T]$.

For a non-empty set $B \subset \mathbb{C}[0, T]$ let

$$I_0^T(B) := \inf_{f \in B} I_0^T(f), \quad I_0^T(\emptyset) := \infty.$$

$(f)_{T,\varepsilon}$ and $(B)_{T,\varepsilon}$ denote ε -neighbourhood in metric ρ_T in space $\mathbb{C}[0, T]$ of $f \in \mathbb{C}[0, T]$ and measurable set $B \subset \mathbb{C}[0, T]$ respectively. The interior and the closure of a measurable set $B \subset \mathbb{C}[0, T]$ is denoted by $(B)_T$ and $[B]_T$ respectively.

Note that lower semi-continuity (2.1) can be written as: *for any* $f \in \mathbb{C}[0, T]$

$$\lim_{\varepsilon \rightarrow 0} I_0^T((f)_{T,\varepsilon}) = I_0^T(f). \quad (2.2)$$

It is obvious that (2.1) and (2.2) are equivalent.

For a function $f \in \mathbb{C}$, $f^{(T)}$ denotes its projection on $\mathbb{C}[0, T]$,

$$f^{(T)} = f^{(T)}(t) := f(t); \quad 0 \leq t \leq T.$$

Denote by $\mathbb{C}_0 \subset \mathbb{C}$ – the class of functions $f \in \mathbb{C}$, such that $f(0) = 0$, $\lim_{t \rightarrow \infty} \frac{f(t)}{1+t^{1+\kappa}} = 0$.

Let now $X_n(t); t \in [0, \infty)$, be a sequence of processes in space \mathbb{C}_0 . We assume the following conditions.

I. For any $T \in (0, \infty)$ processes $X_n^{(T)}$ satisfy LDP in space $\mathbb{C}[0, T]$ with good rate function I_0^T , i.e. for any measurable set $B \subset \mathbb{C}[0, T]$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in B) \leq -I_0^T([B]_T),$$

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in B) \geq -I_0^T((B)_T).$$

Moreover, for any $f \in \mathbb{C}[0, T]$ there is $g = g_f \in \mathbb{C}_0$, such that $g^{(T)} = f$, and for any $U \geq T$ it holds

$$I_0^U(g^{(U)}) = I_0^T(f). \quad (2.3)$$

Condition (2.3) means that one can extend any $f \in \mathbb{C}[0, T]$ for $t > T$ such that the rate function will stay the same. It is natural to call the function $g = g_f$ the most likely extension of f beyond $[0, T]$.

II. For any $r \geq 0$

$$\lim_{T \rightarrow \infty} \sup_{f \in B_r^+} \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} = 0,$$

where

$$B_r^+ := \{f \in \mathbb{C} : \overline{\lim}_{T \rightarrow \infty} I_0^T(f^{(T)}) \leq r\}.$$

III. For any $N < \infty$ and $\varepsilon > 0$ there is $T = T_{N, \varepsilon} < \infty$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(\sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon) \leq -N.$$

Theorem 2.1. Assume conditions **I**, **II** and **III**. Then for any $f \in \mathbb{C}$ there exists

$$\lim_{T \rightarrow \infty} I_0^T(f^{(T)}) =: I(f), \quad (2.4)$$

and it is a good rate function in the space (\mathbb{C}, ρ) . The sequence X_n satisfies LDP in this space with rate function $I(f)$, i.e. for any measurable $B \subset \mathbb{C}$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) \leq -I([B]), \quad (2.5)$$

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) \geq -I((B)), \quad (2.6)$$

where $[B]$, (B) is the closure and the interior of B , respectively, and

$$I(B) := \inf_{f \in B} I(f),$$

with $I(\emptyset) = \infty$.

Note that if for a set B , $I([B]) = I((B)) (= I(B))$, then inequalities (2.5), (2.6) can be replaced by equality

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) = -I(B).$$

Hence the difference

$$D(B) := I((B)) - I([B]) \geq 0$$

describes precision of LDP: the smaller the difference the more precise is the theorem. Theorem 2.6 in [20] gives sufficient conditions for a sequence X_n to satisfy LDP in the space $(\mathbb{C}, \rho^{(P)})$.

We compare our Theorem 2.1 and Theorem 2.6 in [20]. It follows that the rate functions in both theorems are the same. This is because projections $X_n^{(T)}$ on $[0, T]$ satisfy LDP in the space $\mathbb{C}[0, T]$ with uniform metric and rate function $I_0^T(f)$ common for both theorems. Therefore we can compare these theorems by comparing differences

$$D(B) := I((B)) - I([B]) \text{ and } D^{(P)}(B) := I((B)^{(P)}) - I([B]^{(P)}),$$

where $[B]^{(P)}$, $(B)^{(P)}$ are the closure and the interior of B in metric $\rho^{(P)}$, respectively.

As noted earlier, (see also [20]), $\rho^{(P)}(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$, is equivalent to $\rho_T(f_n, f) \rightarrow 0$ for any $T > 0$. Therefore $\rho(f_n, f) \rightarrow 0$ implies $\rho^{(P)}(f_n, f) \rightarrow 0$. It is easy to see that the opposite is not true. Thus

$$[B] \subset [B]^{(P)}, \quad (B)^{(P)} \subset (B),$$

therefore

$$I([B]^{(P)}) \leq I([B]), \quad I((B)) \leq I((B)^{(P)}),$$

so that we have always $D(B) \leq D^{(P)}(B)$. Below we give an example of B satisfying simultaneously

$$I([B]) = I((B)) \in (0, \infty), \quad I([B]^{(P)}) = 0.$$

Hence Theorem 2.1 allows to give “precise” logarithmic asymptotic for $\mathbf{P}(X_n \in B)$, when Theorem 2.6 in [20] does not. We conclude that *LDP in the space (\mathbb{C}, ρ) is more precise than in the space $(\mathbb{C}, \rho^{(P)})$* .

Example 2.1. Consider Wiener process $w = w(t)$ on $[0, \infty)$. Denote

$$w_n = w_n(t) := \frac{1}{\sqrt{n}} w(t), \quad t \geq 0.$$

Since conditions **I**—**III** are easily checked, then LDP follows from Theorem 2.1 with rate function

$$I(f) = \begin{cases} \frac{1}{2} \int_0^\infty (f'(t))^2 dt, & \text{if } f(0) = 0, \text{ } f \text{ is absolutely continuous,} \\ \infty & \text{otherwise.} \end{cases}$$

$$B = \overline{(f_0)_1} := \left\{ g \in \mathbb{C} : \sup_{t \geq 0} \frac{|g(t)|}{1+t} \geq 1 \right\}, \quad f_0 = f_0(t) \equiv 0.$$

Since it is a complement to an open set $(f_0)_1$, it is closed in (\mathbb{C}, ρ) , and therefore

$$I([B]) = I(B) = \inf_{g \in B} I(g).$$

By Cauchy-Bunyakovski inequality

$$1 \leq \sup_{t \geq 0} \frac{|g(t)|}{1+t} = \sup_{t \geq 0} \frac{|\int_0^t g'(s) ds|}{1+t} \leq \sup_{t \geq 0} \left| \int_0^t (g'(s))^2 ds \right|^{1/2} \sup_{t \geq 0} \frac{t^{1/2}}{1+t} = \frac{\sqrt{2I(g)}}{2}.$$

This gives that $I(g) \geq 2$ for all $g \in B$.

Take $f(t) = 2tI(0 \leq t \leq 1) + 2I(t \geq 1)$. It is easy to see that $f \in B$ and $I(f) = 2$. Therefore $I([B]) = I(B) = 2$.

Taking $f_n(t) = (2 + 1/n)tI(0 \leq t \leq 1) + 2 + 1/nI(t \geq 1)$, we can see that $f_n \in (B)$ and $I([B]) = I(B) = I((B)) = 2$. Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(w_n \in B) = -2.$$

Consider now $[B]^{(P)}$, the closure of B in metric $\rho^{(P)}$. By taking $g_n(t) = \frac{t^2}{n}$ it is easy to see that $g_n \in B$ for all n and $\lim_{n \rightarrow \infty} \rho^{(P)}(g_n, f_0) = 0$. Therefore, $f_0 \in [B]^{(P)}$. Therefore $I([B]^{(P)}) = 0$, and the upper bound in Theorem 3.4 for the set B is trivial, which does not allow to find logarithmic asymptotic of the required probability.

3. PROOF OF THEOREM 2.1

For $\varepsilon > 0$ denote by $(f)_\varepsilon$ and $(B)_\varepsilon$ the ε -neighborhood of $f \in \mathbb{C}$, and set $B \subset \mathbb{C}$, respectively.

The proof of the Theorem 2.1 consists of three steps. The first step proves that $I(f)$ is a good rate function in Lemma 3.1. The second step proves the local LDP for X_n in (\mathbb{C}_0, ρ) in Lemma 3.2. The third step proves a weaker form of exponential tightness for X_n in Lemma 3.3.

The upper bound is obtained by Lemmas 3.2 and 3.3, for any measurable set $B \subset \mathbb{C}_0$ and $\varepsilon > 0$ it holds (see e.g. [4], Theorem 3.1)

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) \leq -I((B)_\varepsilon).$$

As it is known (see e.g. [4], Lemma 2.1), that a good rate function $I(f)$ satisfies

$$\lim_{\varepsilon \rightarrow 0} I((B)_\varepsilon) = I([B]),$$

the upper bound (2.5) is proved. Lower bound (2.6) follows from (3.5) of Lemma 3.2.

Lemma 3.1. *The rate function $I(f)$ (defined in (2.4)) is a good rate function, i.e. for any $r \geq 0$ the set*

$$B_r := \{f \in \mathbb{C} : I(f) \leq r\}$$

is a compact in \mathbb{C} and

$$\lim_{\varepsilon \rightarrow 0} I((f)_\varepsilon) = I(f). \quad (3.1)$$

Proof. First we show that the limit exists. It is known (see e.g. [4], Theorem 3.1 or Lemma 1.3), that LDP implies local LDP: for any $f \in \mathbb{C}[0, T]$

$$-I_0^T(f) \geq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f)_{T,\varepsilon}) \geq \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f)_{T,\varepsilon}) \geq -I_0^T(f).$$

For $U \geq T$, with obvious notations, we have for $f \in \mathbb{C}$

$$\{X_n^{(U)} \in (f^{(U)})_{U,\varepsilon}\} \subset \{X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}\},$$

therefore

$$\begin{aligned} -I_0^T(f^{(T)}) &\geq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}) \\ &\geq \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(U)} \in (f^{(U)})_{U,\varepsilon}) \geq -I_0^U(f^{(U)}). \end{aligned}$$

Thus we established that $I_0^T(f^{(T)})$ is non-decreasing in T , and (2.4) follows.

Next, we show lower semi-continuity (3.1), that is if $f_n \rightarrow f$, then

$$\underline{\lim}_{n \rightarrow \infty} I(f_n) \geq I(f). \quad (3.2)$$

For any $N < \infty$, $\varepsilon > 0$ there is $T = T_{N,\varepsilon} < \infty$ such that

$$I_0^T(f) \geq \min\{I(f), N\} - \varepsilon.$$

Since $f_n \rightarrow f$, $\rho_T(f_n, f) \rightarrow 0$. The rate function $I_0^T(f)$ is lower semi-continuous in $(\mathbb{C}[0, T], \rho_T)$ (due to condition **I**), therefore

$$\underline{\lim}_{n \rightarrow \infty} I(f_n) \geq \underline{\lim}_{n \rightarrow \infty} I_0^T(f_n) \geq I_0^T(f) \geq \min\{I(f), N\} - \varepsilon.$$

Since $N < \infty$ and $\varepsilon > 0$ are arbitrary, the latter implies (3.2).

We show next that the set B_r is completely bounded. For any $\varepsilon > 0$ due to condition **II** there is $T = T_r < \infty$ such that for any $f \in B_r$

$$\sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon. \quad (3.3)$$

Denote

$$B_r^{(T)} := \{f^{(T)} : f \in B_r\},$$

so that

$$B_r^{(T)} \subset B_{T,r},$$

where we recall that $B_{T,r} := \{f \in \mathbb{C}[0, T] : I_0^T(f) \leq r\}$.

Since by **I** the set $B_{T,r}$ is a compact in $\mathbb{C}[0, T]$, it is possible to find finite ε -net:

$$B_r^{(T)} \subset B_{T,r} \subset \cup_{i=1}^M (f_i)_{T,\varepsilon}.$$

Now for $f \in \mathbb{C}[0, T]$ define $f^{(T+)} \in \mathbb{C}_0$ as

$$f^{(T+)}(t) := \begin{cases} f(t), & \text{if } 0 \leq t \leq T, \\ f(T), & \text{if } t \geq T \end{cases}$$

For any $f \in B_r$ there is $i \in \{1, \dots, M\}$ such that

$$\sup_{0 \leq t \leq T} \frac{|f(t) - f_i^{(T+)}(t)|}{1 + t^{1+\kappa}} < \varepsilon < 3\varepsilon.$$

We have for this i due to (3.3)

$$\sup_{t \geq T} \frac{|f(t) - f_i^{(T+)}(t)|}{1 + t^{1+\kappa}} \leq \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} + \sup_{t \geq T} \frac{|f(T)|}{1 + t^{1+\kappa}} + \sup_{t \geq T} \frac{|f(T) - f_i^{(T+)}(T)|}{1 + t^{1+\kappa}} \leq 3\varepsilon,$$

therefore the collection $\{f_1^{(T+)}, \dots, f_M^{(T+)}\}$ represents a 3ε -net in the set B_r . Thus we have shown that the set B_r is completely bounded in \mathbb{C}_0 .

From lower semi-continuity of $I(f)$, established earlier, it follows that B_r is closed in \mathbb{C}_0 . Since a closed completely bounded subset of a Polish space is a compact (see [16], Theorem 3, p. 109), we have shown that B_r is a compact in \mathbb{C}_0 , thus completing the proof of Lemma 3.1. \square

Lemma 3.2. For any $f \in \mathbb{C}_0$, $\varepsilon > 0$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \leq -I((f)_{2\varepsilon}), \quad (3.4)$$

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \geq -I(f). \quad (3.5)$$

Proof. (i). First we prove the lower bound (3.5) as it is also used in the proof of the upper bound. If $I(f) = \infty$, then (3.5) is trivially satisfied. Let now $I(f) < \infty$. For any $T \in (0, \infty)$ there holds the inclusion

$$\{X_n \in (f)_\varepsilon\} \supset A_n(T) \cap B_n(T) \cap D(T),$$

where

$$A_n(T) := \left\{ \sup_{0 \leq t \leq T} \frac{|X_n(t) - f(t)|}{1 + t^{1+\kappa}} < \varepsilon \right\}, \quad B_n(T) := \left\{ \sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} < \varepsilon/4 \right\},$$

$$D(T) := \left\{ \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon/2 \right\}.$$

For a large T the event $D(T)$ is a certainty (due to $I(f) < \infty$). Therefore there exists $T_0 < \infty$, such that for all $T \geq T_0$ it holds that

$$\mathbf{P}(X_n \in (f)_\varepsilon) \geq \mathbf{P}(A_n(T) \cap B_n(T)) \geq \mathbf{P}(A_n(T)) - \mathbf{P}(\overline{B_n(T)}), \quad (3.6)$$

where $\overline{B_n(T)}$ is a complement of $B_n(T)$. Due to condition **III** there is $T \geq T_0$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(\overline{B_n(T)}) \leq -2I(f), \quad (3.7)$$

and for this T due to **I** we have

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(A_n(T)) \geq -I_0^T(f^{(T)}) \geq -I(f). \quad (3.8)$$

(3.5) now follows from (3.6) by using (3.7), (3.8).

(ii). Now we prove the upper bound (3.4). It is obvious that for any $T \in (0, \infty)$

$$\mathbf{P}(X_n \in (f)_\varepsilon) \leq \mathbf{P}(X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}),$$

where we recall that $(f)_{T,\varepsilon}$ denote ε -neighbourhood in metric ρ_T in space $\mathbb{C}[0, T]$ of $f \in \mathbb{C}[0, T]$.

Due to condition **I** for any $\delta > 0$

$$\begin{aligned} L(\varepsilon) := \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) &\leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}) \\ &\leq -I_0^T([(f^{(T)})_{T,\varepsilon}]_T) \leq -I_0^T((f^{(T)})_{T,\varepsilon+\delta}). \end{aligned}$$

For any $T \in (0, \infty)$ and chosen ε and δ , in this way we have the inequality

$$L(\varepsilon) \leq -I_0^T((f^{(T)})_{T,\varepsilon+\delta}). \quad (3.9)$$

Choose now $T < \infty$ so large, that simultaneously the following holds:

$$\sup_{t \geq T} \frac{|f(t)|}{1+t^{1+\kappa}} < \delta; \quad (3.10)$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in R(T, \varepsilon)) \leq -N, \quad (3.11)$$

where $N < \infty$ is arbitrary, and

$$R(T, \varepsilon) := \left\{ g \in \mathbb{C}_0 : \sup_{t \geq T} \frac{|g(t)|}{1+t^{1+\kappa}} > \varepsilon \right\}.$$

Denote

$$((f^{(T)})_{T,\varepsilon+\delta})^{(T+)} := \{g \in \mathbb{C}_0 : g^{(T)} \in (f^{(T)})_{T,\varepsilon+\delta}\}.$$

Next we show that

$$I_0^T(B) = I(B^{(T+)}), \quad (3.12)$$

where for $B \subset \mathbb{C}[0, T]$

$$B^{(T+)} := \{g \in \mathbb{C}_0 : g^{(T)} \in B\}.$$

Indeed, for any $\varepsilon > 0$ let $f \in B$ be such that

$$I_0^T(f) \leq I_0^T(B) + \varepsilon.$$

Then due to (2.3) in condition **I** there is $g \in \mathbb{C}_0$ such that $g^{(T)} = f$ (consequently $g \in B^{(T+)}$) with $I(g) = I_0^T(f)$. Therefore

$$I_0^T(B) + \varepsilon \geq I_0^T(f) = I(g) \geq I(B^{(T+)}).$$

Since $\varepsilon > 0$ is arbitrary,

$$I_0^T(B) \geq I(B^{(T+)}). \quad (3.13)$$

Let now $g \in B^{(T+)}$ such that

$$I(g) \leq I(B^{(T+)}) + \varepsilon.$$

Then $g^{(T)} \in B$ with $I_0^T(g^{(T)}) \leq I(g)$. Therefore

$$I(B^{(T+)}) + \varepsilon \geq I(g) \geq I_0^T(g^{(T)}) \geq I_0^T(B),$$

and

$$I_0^T(B) \leq I(B^{(T+)}). \quad (3.14)$$

Inequalities (3.13), (3.14) now prove equality (3.12).

Due to (3.12) we have

$$I_0^T((f^{(T)})_{T,\varepsilon+\delta}) = I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}),$$

therefore due to (3.9)

$$L(\varepsilon) \leq -I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}). \quad (3.15)$$

Take an arbitrary $g \in ((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}$. Then either

$$\sup_{t \geq T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} < \varepsilon + 2\delta,$$

and then

$$g \in (f)_{\varepsilon+2\delta}; \quad (3.16)$$

or

$$\sup_{t \geq T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} \geq \varepsilon + 2\delta, \quad (3.17)$$

and then

$$\sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} \geq \varepsilon + \delta, \quad (3.18)$$

and

$$g \in R(T, \varepsilon). \quad (3.19)$$

To clarify deduction of (3.18) from (3.17), note that if the inequality (3.18) is not true, then the opposite holds

$$\sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} < \varepsilon + \delta,$$

and due to (3.10)

$$\sup_{t \geq T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} \leq \sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} + \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon + \delta + \delta = \varepsilon + 2\delta,$$

which contradicts (3.17). We have proved (see (3.16) and (3.19)), that

$$((f^{(T)})_{T,\varepsilon+\delta})^{(T+)} \subset (f)_{\varepsilon+2\delta} \cup R(T, \varepsilon).$$

From the latter we obtain

$$I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}) \geq \min\{I((f)_{\varepsilon+2\delta}), I(R(T, \varepsilon))\}. \quad (3.20)$$

Further, due to (3.11)

$$-N \geq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in R(T, \varepsilon)) \geq \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in R(T, \varepsilon)) \geq -I(R(T, \varepsilon)),$$

where the last inequality for an open set $R(T, \varepsilon)$ follows from the established lower bound (3.5). Therefore

$$I(R(T, \varepsilon)) \geq N,$$

and, in view of (3.20),

$$I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}) \geq \min\{I((f)_{\varepsilon+2\delta}), N\}.$$

Going back to (3.15), we obtain the inequality

$$L(\varepsilon) \leq -\min\{I((f)_{\varepsilon+2\delta}), N\},$$

in which $\delta > 0$ and $N < \infty$ are arbitrary. Taking $2\delta = \varepsilon$ and sending N to ∞ , we obtain the required upper bound

$$L(\varepsilon) \leq -I((f)_{2\varepsilon}).$$

Lemma 3.2 is now proved. \square

Local LDP for $\{X_n\}$ in \mathbb{C}_0 follows from 3.2, and is stated as a corollary.

Corollary 3.1. *For any $f \in \mathbb{C}_0$*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) &\leq -I(f), \\ \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) &\geq -I(f). \end{aligned}$$

Next result proves a weaker form of exponential tightness: for any N there is a completely bounded set K_N in (\mathbb{C}_0, ρ) such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \notin K) \leq -N.$$

Lemma 3.3. *For any $N < \infty$ and $\varepsilon > 0$ there is a finite collection of $g_1, \dots, g_M \in \mathbb{C}_0$ such that*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \notin \cup_{i=1}^M (g_i)_\varepsilon) \leq -N.$$

Proof. Denote by

$$R_T(\varepsilon) := \left\{ f \in \mathbb{C}_0 : \sup_{t \geq T} \frac{|f(t)|}{1+t^{1+\kappa}} \leq \varepsilon \right\}.$$

Then due to condition **III** there is $T < \infty$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \notin R_T(\varepsilon)) \leq -N. \quad (3.21)$$

For this T due to condition **I** the process $X_n^{(T)}$ satisfies LDP in the space $\mathbb{C}[0, T]$. Therefore for a chosen N by a theorem of Puhalskii (see [20] **page or Theorem number**) there is a compact $K \subset \mathbb{C}[0, T]$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \notin K) \leq -N.$$

For a given $\varepsilon > 0$ take a finite ε -net $f_1, \dots, f_M \in \mathbb{C}[0, T]$ in K :

$$K \subset \cup_{i=1}^M (f_i)_{T, \varepsilon}.$$

Then

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \notin \cup_{i=1}^M (f_i)_{T, \varepsilon}) \leq -N. \quad (3.22)$$

Denote for all $i = 1, \dots, M$

$$g_i(t) := \begin{cases} f_i(t), & \text{if } 0 \leq t \leq T, \\ f_i(T), & \text{if } t \geq T. \end{cases}$$

Define the set $\mathcal{M}_\varepsilon := \{i \in \{1, \dots, M\} : \frac{|f_i(T)|}{1+T^{1+\kappa}} \leq 2\varepsilon\}$. Then

$$P := \mathbf{P}(X_n \notin \cup_{i=1}^M (g_i)_{3\varepsilon}) \leq \mathbf{P}(X_n \notin R_T(\varepsilon)) + \mathbf{P}(X_n \in R_T(\varepsilon), X_n^{(T)} \notin \cup_{i=1}^M (f_i)_{T, \varepsilon}) +$$

$$\mathbf{P}(X_n \in R_T(\varepsilon), X_n^{(T)} \in \cup_{i=1}^M (f_i)_{T,\varepsilon}, X_n \notin \cup_{i=1}^M (g_i)_{3\varepsilon}) =: P_1 + P_2 + P_3.$$

We bound P_3 as follows:

$$P_3 \leq \sum_{i \in \mathcal{M}_\varepsilon} \mathbf{P} \left(\sup_{t \geq T} \frac{|X_n(t) - f_i(T)|}{1 + t^{1+\kappa}} > 3\varepsilon \right) \leq M \mathbf{P} \left(\sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon \right) \leq M \mathbf{P}(X_n \notin R_T(\varepsilon)).$$

Since

$$P_1 \leq \mathbf{P}(X_n \notin R_T(\varepsilon)),$$

we obtain

$$P \leq \mathbf{P}(X_n^{(T)} \notin \cup_{i=1}^M (f_i)_{T,\varepsilon}) + (M+1) \mathbf{P}(X_n \notin R_T(\varepsilon)). \quad (3.23)$$

Using bounds (3.21) and (3.22) with (3.23), we obtain the required inequality

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}(X_n \notin \cup_{i=1}^M (g_i)_{3\varepsilon}) \leq -N.$$

Lemma 3.3 is now proved. \square

4. LARGE DEVIATIONS FOR RANDOM WALKS

4.1. Large Deviation Principle for Random Walks on half-line. Let ξ be a non-degenerate random variable satisfying the following condition

$[\mathbf{C}_\infty]$. For any $\lambda \in \mathbb{R}$

$$\mathbf{E} e^{\lambda \xi} < \infty.$$

Denote

$$\Lambda(\alpha) := \sup_{\lambda} \{\lambda \alpha - A(\lambda)\}, \quad A(\lambda) := \ln \mathbf{E} e^{\lambda \xi},$$

the deviation function of ξ . It is a convex non-negative lower-semicontinuous function with a single zero at $\alpha = \mathbf{E} \xi$, (see e.g. [2] or [5]).

Denote

$$S_0 := 0, \quad S_k := \xi_1 + \cdots + \xi_k \text{ for } k \geq 1,$$

where $\{\xi_n\}$ is a sequence of i.i.d. copies of ξ . Consider a random piece-wise linear function $s_n = s_n(t) \in \mathbb{C}$, going through the nodes

$$\left(\frac{k}{n}, \frac{S_k}{x} \right), \quad k = 0, 1, \dots,$$

where $x = x(n)$ is a fixed sequence of positive constants such that $x \sim n$ as $n \rightarrow \infty$. The rate function corresponding to the process s_n is defined as

$$I(f) := \begin{cases} \int_0^\infty \Lambda(f'(t)) dt, & f(0) = 0, f \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 4.1. Assume $[\mathbf{C}_\infty]$. Then s_n satisfies LDP in space $(\mathbb{C}, \rho_\kappa)$ for $\kappa = 0$ with rate function I .

Proof. Without loss of generality we can take $\mathbf{E}\xi = 0$. This is because the deviation function for $\xi^{(0)} := \xi - a$ is given by $\Lambda^{(0)}(\alpha) = \Lambda(\alpha + a)$. (superscript $^{(0)}$ denotes quantities for the centered random variable). Therefore the rate function for $s_n^{(0)}$, is given by $I^{(0)}(f) = I(f + e_a)$, where $e_a = e_a(t) := at$; $t \geq 0$. Clearly, $s_n = s_n^{(0)} + e_a$, $\mathbf{P}(s_n \in B) = \mathbf{P}(s_n^{(0)} \in B - e_a)$, where $B - e_a := \{f - e_a : f \in B\}$. It is obvious that $[B - e_a] = [B] - e_a$, $(B - e_a) = (B) - e_a$, implying $I^{(0)}([B - e_a]) = I([B])$, $I^{(0)}((B - e_a)) = I((B))$. Hence the LDP for $s_n^{(0)}$ with rate function $I^{(0)}$ implies LDP for s_n with rate function I .

The rest of the proof consists in checking conditions **I** – **III** of Theorem 2.1. Condition **I** follows from the LDP for s_n in $\mathbb{C}[0, 1]$ (see [5], Theorem 9 or [4], Section 6.2).

Proof of **II**. By $[\mathbf{C}_\infty]$, with $\mathbf{E}\xi = 0$ it follows that there exists a non-decreasing continuous function $h(t)$; $t \geq 0$, such that for some $\delta > 0$, $h(t) = \delta t$, if $0 \leq t \leq 1$, $\lim_{t \rightarrow \infty} h(t) = \infty$, and that for all $\alpha \in \mathbb{R}$ the following inequality holds

$$\Lambda(\alpha) \geq h(|\alpha|)|\alpha|. \quad (4.1)$$

Indeed, for $\alpha \rightarrow 0$ (see e.g. [5], p.21)

$$\Lambda(\alpha) \sim \frac{\alpha^2}{2\sigma^2}; \quad (4.2)$$

for any $\lambda > 0$, $\alpha > 0$

$$\Lambda(\alpha) \geq \lambda\alpha - A(\lambda), \quad \Lambda(-\alpha) \geq \lambda\alpha - A(-\lambda).$$

Therefore

$$\liminf_{|\alpha| \rightarrow \infty} \frac{\Lambda(\alpha)}{|\alpha|} \geq \lambda,$$

so that

$$\liminf_{|\alpha| \rightarrow \infty} \frac{\Lambda(\alpha)}{|\alpha|} = \infty. \quad (4.3)$$

(4.1) follows from (4.2) and (4.3).

Denote by

$$f_T(t) := t \frac{f(T)}{T}, \quad t \in [0, T].$$

The function f_T “straightens” function f on $[0, T]$:

$$I_0^T(f) \geq I_0^T(f_T) = \int_0^T \Lambda\left(\frac{f(T)}{T}\right) dt = T \Lambda\left(\frac{f(T)}{T}\right).$$

Therefore by (4.1) for $f \in B_r$

$$r \geq T \Lambda\left(\frac{f(T)}{T}\right) \geq T \frac{|f(T)|}{T} h\left(\frac{|f(T)|}{T}\right),$$

so that

$$\frac{|f(T)|}{T} \leq \frac{r}{Th\left(\frac{|f(T)|}{T}\right)}. \quad (4.4)$$

Let $c := \sqrt{\frac{r}{\delta}}$, and T be such that $\frac{c}{\sqrt{T}} \leq 1$. Assume that

$$\frac{|f(T)|}{T} > \frac{c}{\sqrt{T}}.$$

Then it follows from (4.4)

$$\frac{|f(T)|}{T} \leq \frac{r}{Th(\frac{c}{\sqrt{T}})} = \frac{r}{T\delta\frac{c}{\sqrt{T}}} = \frac{c}{\sqrt{T}},$$

which is a contradiction. Thus for $T \geq c^2 = \frac{r}{\delta}$ it holds

$$|f(T)| \leq \sqrt{\frac{r}{\delta}}\sqrt{T}.$$

Clearly, $B_r = B_r^+$. Therefore we have proved

$$\sup_{f \in B_r^+} \sup_{t \geq T} \frac{|f(t)|}{1+t} \leq \sqrt{\frac{r}{\delta}} \frac{\sqrt{T}}{1+T} \leq \sqrt{\frac{r}{\delta}} \frac{1}{\sqrt{T}}.$$

Condition **II** now follows.

Check now condition **III**. For $T_n := \max\{\frac{k}{n} \leq T : k = 1, 2, \dots\}$, we have

$$\begin{aligned} \mathbf{P}\left(\sup_{t \geq T} \frac{|s_n(t)|}{1+t} > \varepsilon\right) &\leq \mathbf{P}\left(\sup_{t \geq T_n} \frac{|s_n(t)|}{1+t} > \varepsilon\right) \\ &\leq \mathbf{P}\left(\sup_{t \geq T_n} \frac{|s_n(t) - s_n(T_n)|}{1+t} > \varepsilon/2\right) + \mathbf{P}\left(\sup_{t \geq T_n} \frac{|s_n(T_n)|}{1+t} > \varepsilon/2\right) \\ &= \mathbf{P}\left(\sup_{u \geq 0} \frac{|s_n(u)|}{1+T_n+u} > \varepsilon/2\right) + \mathbf{P}\left(\sup_{t \geq T_n} \frac{|s_n(T_n)|}{1+T_n} > \varepsilon/2\right) \\ &\leq \mathbf{P}\left(\sup_{k \geq 1} \frac{|s_n(\frac{k}{n})|}{T + \frac{k}{n}} > \varepsilon/2\right) + \mathbf{P}\left(\frac{|s_n(T_n)|}{T_n} > \varepsilon/2\right) =: P_1(n) + P_2(n). \end{aligned}$$

To bound $P_1(n)$ use the exponential Chebyshev's (Chernoff's) inequality (see e.g. [2] or [5]):

$$\begin{aligned} P_1(n) &\leq \sum_{k \geq 1} \mathbf{P}\left(\frac{|S_k|}{x(T + \frac{k}{n})} > \varepsilon/2\right) \\ &\leq \sum_{k \geq 1} \mathbf{P}\left(\frac{S_k}{x(T + \frac{k}{n})} > \varepsilon/2\right) + \sum_{k \geq 1} \mathbf{P}\left(\frac{S_k}{x(T + \frac{k}{n})} < -\varepsilon/2\right) \\ &\leq \sum_{k \geq 1} e^{-k\Lambda(R)} + \sum_{k \geq 1} e^{-k\Lambda(-R)}, \end{aligned}$$

where $R := \frac{x}{k}(T + \frac{k}{n})$. Since for all n and T large enough

$$R \geq \varepsilon/4, \quad kR \geq T\varepsilon/4 + k\varepsilon/4,$$

we have due to (4.1) for $\varepsilon/4 \in (0, 1)$

$$k\Lambda(\pm R) \geq kRh(R) \geq (T\varepsilon/4 + k\varepsilon/4)\delta\varepsilon/4 = T\frac{\delta\varepsilon^2}{16} + k\frac{\delta\varepsilon^2}{16}.$$

Therefore

$$P_1(n) \leq 2e^{-T\delta_1} \sum_{k \geq 1} e^{-k\delta_1} = C_1 e^{-T\delta_1},$$

where $\delta_1 := \frac{\delta \varepsilon^2}{16}$, $C_1 := 2 \frac{e^{-\delta_1}}{1 - e^{-s_1}}$.

Similarly we obtain the bound

$$P_2(n) \leq C_2 e^{-T\delta_2}$$

for some $\delta_2 > 0$, $C_2 < \infty$. Hence condition **III** holds and the proof is complete. \square

4.2. Moderate Deviation Principle for Random Walks on half-line. Let random piece-wise linear function $s_n = s_n(\cdot) \in \mathbb{C}$ be defined as before by the sums S_k of independent random variables distributed as ξ . Let ξ have zero mean $\mathbf{E}\xi = 0$ and assume Cramer's condition

[C₀]. For some $\delta > 0$

$$\mathbf{E} e^{\delta|\xi|} < \infty.$$

Let a sequence $x = x(n)$, used in the construction of s_n , satisfy

$$\frac{x}{\sqrt{n}} \rightarrow \infty, \quad \frac{x}{n} \rightarrow 0 \text{ if } n \rightarrow \infty.$$

The rate function for s_n is defined as

$$I_0(f) := \begin{cases} \frac{1}{2\sigma^2} \int_0^\infty (f'(t))^2 dt, & \text{if } f(0) = 0, \text{ } f \text{ is absolutely continuous} \\ \infty & \text{otherwise,} \end{cases}$$

where $\sigma^2 := \mathbf{E}\xi^2$.

Theorem 4.2. Let $\mathbf{E}\xi = 0$ and condition **[C₀]** holds. Then s_n satisfies LDP with speed $\frac{x^2}{n}$ and rate function I_0 in space $(\mathbb{C}, \rho_\kappa)$ with $\kappa = 0$, i.e. for any measurable set $B \subset \mathbb{C}$

$$\overline{\lim}_{n \rightarrow \infty} \frac{n}{x^2} \ln \mathbf{P}(s_n \in B) \leq -I_0([B]),$$

$$\underline{\lim}_{n \rightarrow \infty} \frac{n}{x^2} \ln \mathbf{P}(s_n \in B) \geq -I_0((B)).$$

Similarly to the proof of Lemma 3.1, the proof of Theorem 3.2 consists in checking conditions **I**–**III**, replacing n by $\frac{x^2}{n}$. In all other details the proof is the same.

Condition **I** is verified with help of [18] (Theorem 1) or [6] (Theorem 2.2). Condition **II** is obvious. Only condition **III** requires a clarification, which is done by using the following form of Kolmogorov's inequality ([1], p. 295, lemma 11.2.1):

Lemma 4.1. For any $x \geq 0$, $y \geq 0$, $n \geq 1$

$$\mathbf{P}(\max_{1 \leq m \leq n} |S_m| \geq x + y) \leq \frac{\mathbf{P}(|S_n| \geq x)}{\min_{1 \leq m \leq n} \mathbf{P}(|S_m| \leq y)}.$$

Proof. An upper bound for

$$P := \mathbf{P} \left(\sup_{t \geq T} \frac{|s_n(t)|}{1+t} \geq \varepsilon \right)$$

is obtained by using

$$\begin{aligned} P &\leq \sum_{K \geq T} \mathbf{P} \left(\sup_{K \leq t \leq K+1} |s_n(t)| \geq K\varepsilon \right) \\ &\leq \sum_{K \geq T} \mathbf{P} \left(\sup_{K \leq t \leq K+1} |s_n(t) - s_n(K)| \geq K\varepsilon/2 \right) + \sum_{K \geq T} \mathbf{P}(|s_n(K)| \geq K\varepsilon/2), \end{aligned}$$

so that

$$P \leq \sum_{K \geq T} P_1(K, n) + \sum_{K \geq T} P_2(K, n), \quad (4.5)$$

where

$$P_1(K, n) := \mathbf{P} \left(\sup_{0 \leq t \leq 1} |s_n(u)| \geq K\varepsilon/2 \right), \quad P_2(K, n) := \mathbf{P}(|s_n(K)| \geq K\varepsilon/2).$$

We bound $P_2(K, n)$ using exponential Chebyshev's inequality:

$$P_2(K, n) = \mathbf{P} \left(\frac{|S_{nK}|}{nK} \geq \frac{x\varepsilon}{2n} \right) \leq e^{-nK\Lambda(\frac{x\varepsilon}{2n})} + e^{-nK\Lambda(-\frac{x\varepsilon}{2n})}.$$

Since for all n large enough $\frac{x\varepsilon}{2n} \leq 1$, then by (4.1)

$$nK\Lambda \left(\pm \frac{x\varepsilon}{2n} \right) \geq \frac{x^2}{n} K\delta_1, \quad \delta_1 := \frac{\delta\varepsilon^2}{4}.$$

Therefore

$$\sum_{K \geq T} P_2(K, n) \leq 2 \frac{e^{-\frac{x^2}{n} T\delta_1}}{1 - e^{-\frac{x^2}{n} \delta_1}}. \quad (4.6)$$

Bound now $P_1(K, n)$ using Kolmogorov's inequality (Lemma 3.2) and exponential Chebyshev's inequality (Chernoff)

$$P_1(K, n) = \mathbf{P} \left(\max_{1 \leq m \leq n} \frac{|S_m|}{xK} \geq \varepsilon/4 + \varepsilon/4 \right) \leq \frac{1}{c} \mathbf{P} \left(\frac{|S_n|}{xK} \geq \varepsilon/4 \right),$$

where

$$c := \min_{1 \leq m \leq n} \mathbf{P} \left(\frac{|S_m|}{xK} < \varepsilon/4 \right).$$

Since

$$c = \min_{1 \leq m \leq n} \mathbf{P} \left(\frac{|S_m|}{xK} < \varepsilon/4 \right) \geq \min_{1 \leq m \leq n} \mathbf{P} \left(\frac{|S_m|}{\sqrt{m}} < \frac{xT}{\sqrt{n}} \varepsilon/4 \right) \rightarrow 1$$

as $n \rightarrow \infty$, for n large enough

$$P_1(K, n) \leq 2\mathbf{P} \left(\frac{|S_n|}{xK} \geq \varepsilon/4 \right) \leq 2e^{-n\Lambda(\frac{xK}{n}\varepsilon/4)} + 2e^{-n\Lambda(-\frac{xK}{n}\varepsilon/4)}.$$

By (4.1) for large enough n and some $\delta_1 > 0$ we have

$$n\Lambda \left(\pm \frac{xK}{n} \varepsilon/4 \right) \geq \frac{x^2}{n} K\delta_1,$$

therefore

$$P_1(K, n) \leq 4e^{-\frac{x^2}{n}K\delta_1},$$

$$\sum_{K \geq T} P_1(K, n) \leq 4 \frac{e^{-\frac{x^2}{n}T\delta_1}}{1 - e^{-\frac{x^2}{n}\delta_1}}. \quad (4.7)$$

Applying (4.6), (4.7) to (4.5), we have for $T \geq \frac{N}{\delta_1}$ the required inequality in **III**:

$$\overline{\lim}_{n \rightarrow \infty} \frac{n}{x^2} \ln \mathbf{P} \leq -N.$$

Thus condition **III** is proved. \square

5. LARGE DEVIATIONS FOR DIFFUSION PROCESSES ON HALF-LINE

5.1. Zero drift. Consider a stochastic process $X_n(t)$, $t \geq 0$, defined on the stochastic basis $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbf{P})$ that is an Itô integral with respect to Wiener process $w(t)$.

$$X_n(t) = x_0 + \frac{1}{\sqrt{n}} \int_0^t \sigma_n(\omega, s) dw(s),$$

where $\sigma_n(\omega, t)$ is \mathfrak{F}_t -adapted and such that the Itô integral is defined.

Lemma 5.1. *Let for some $\lambda > 0$ and all $t \geq 0$, $n \geq 0$*

$$\sigma_n^2(\omega, t) \leq \lambda \text{ a.s.} \quad (5.1)$$

Then for any $N < \infty$ and $\varepsilon > 0$ there exists $T = T_{N, \varepsilon} < \infty$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P} \left(\sup_{t \geq T} \frac{|X_n(t)|}{1+t} > \varepsilon \right) \leq -N. \quad (5.2)$$

Proof. For $T > 1 \vee (\frac{2|x_0|}{\varepsilon} - 1)$ we have

$$\begin{aligned} \mathbf{P} \left(\sup_{t \geq T} \frac{|X_n(t)|}{1+t} > \varepsilon \right) &\leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{1+t} \left| \frac{1}{\sqrt{n}} \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{\varepsilon}{2} \right) \\ &\leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{t} \left| \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{\sqrt{n}\varepsilon}{2} \right) \\ &= \mathbf{P} \left(\bigcup_{r=1}^{\infty} \left\{ \sup_{t \in [Tr, T(r+1))} \frac{1}{t} \left| \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{\sqrt{n}\varepsilon}{2} \right\} \right) \\ &\leq \sum_{r=1}^{\infty} \mathbf{P} \left(\sup_{t \in [0, T(r+1)]} \left| \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{Tr\sqrt{n}\varepsilon}{2} \right) =: \sum_{r=1}^{\infty} P_r. \end{aligned}$$

We bound P_r from above as follows. For any $c > 0$ we have

$$\begin{aligned} P_r &= \mathbf{P}\left(\sup_{t \in [0, T(r+1)]} \exp\left\{\left|c \int_0^t \sigma_n(\omega, s) dw(s)\right|\right\} > \exp\left\{\frac{cTr\sqrt{n}\varepsilon}{2}\right\}\right) \\ &\leq \mathbf{P}\left(\sup_{t \in [0, T(r+1)]} \exp\left\{c \int_0^t \sigma_n(\omega, s) dw(s)\right\} > \exp\left\{\frac{cTr\sqrt{n}\varepsilon}{2}\right\}\right) \\ &\quad + \mathbf{P}\left(\sup_{t \in [0, T(r+1)]} \exp\left\{-c \int_0^t \sigma_n(\omega, s) dw(s)\right\} > \exp\left\{\frac{cTr\sqrt{n}\varepsilon}{2}\right\}\right) =: P_{r,1} + P_{r,2}. \end{aligned}$$

We proceed to bound $\mathbf{P}_{r,1}$. For ease of notation we drop arguments in $\sigma_n(\omega, t)$.

Using (5.1) we have

$$\begin{aligned} P_{r,1} &= \mathbf{P}\left(\sup_{t \in [0, T(r+1)]} \exp\left\{c \int_0^t \sigma_n dw(s) \pm \frac{c^2}{2} \int_0^t \sigma_n^2 ds\right\} > \exp\left\{\frac{cTr\sqrt{n}\varepsilon}{2}\right\}\right) \\ &\leq \mathbf{P}\left(\sup_{t \in [0, T(r+1)]} \exp\left\{c \int_0^t \sigma_n dw(s) - \frac{c^2}{2} \int_0^t \sigma_n^2 ds\right\} > \exp\left\{\frac{cTr\sqrt{n}\varepsilon - c^2\lambda T(r+1)}{2}\right\}\right). \end{aligned}$$

By Doob's martingale inequality for

$$M(t) = \exp\left\{c \int_0^t \sigma_n(\omega, s) dw(s) - \frac{c^2}{2} \int_0^t \sigma_n^2(\omega, s) ds\right\},$$

we have

$$P_{r,1} \leq \frac{\mathbf{E}M(t)}{\exp\left\{\frac{cTr\sqrt{n}\varepsilon - c^2\lambda T(r+1)}{2}\right\}} = \exp\left\{-\frac{cTr\sqrt{n}\varepsilon - c^2\lambda T(r+1)}{2}\right\}.$$

Taking $c = \frac{\sqrt{n}\varepsilon Tr}{2\lambda T(r+1)}$ we obtain

$$P_{r,1} \leq \exp\left\{-\frac{(Tr)^2 n \varepsilon^2}{8\lambda T(r+1)}\right\} \leq \exp\left\{-\frac{Trn\varepsilon^2}{16\lambda}\right\}. \quad (5.3)$$

$P_{r,2}$ is bounded in exactly the same way. Collecting the terms it now follows

$$\begin{aligned} \mathbf{P}\left(\sup_{t \geq T} \frac{|X_n(t)|}{1+t} > \varepsilon\right) &\leq 2 \sum_{r=1}^{\infty} \exp\left\{-\frac{Trn\varepsilon^2}{16\lambda}\right\} \\ &= \frac{2 \exp\left\{-\frac{Tn\varepsilon^2}{16\lambda}\right\}}{1 - \exp\left\{-\frac{Tn\varepsilon^2}{16\lambda}\right\}} \leq 4 \exp\left\{-\frac{Tn\varepsilon^2}{16\lambda}\right\}. \end{aligned} \quad (5.4)$$

Using inequality (5.4) we have

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}\left(\sup_{t \geq T} \frac{|X_n(t)|}{1+t} > \varepsilon\right) \leq -\frac{T\varepsilon^2}{16\lambda}.$$

It follows now that for $T > 1 \vee \left(\frac{2|x_0|}{\varepsilon} - 1\right) \vee \frac{16\lambda N}{\varepsilon^2}$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}\left(\sup_{t \geq T} \frac{|X_n(t)|}{1+t} > \varepsilon\right) \leq -N,$$

which proves (5.2). \square

Let now X_n solve a stochastic differential equation (SDE)

$$X_n(t) = x_0 + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s) \quad (5.5)$$

on half-line $[0, \infty)$.

Theorem 5.1. *Let $\sigma(x)$ be a measurable function of real argument x , such that for some $\lambda \geq 1$ and all $x \in R$*

$$\frac{1}{\lambda} \leq \sigma^2(x) \leq \lambda. \quad (5.6)$$

Let the Lebesgue measure of discontinuities of σ be zero. Then $\{X_n\}$ satisfies LDP in space $(\mathbb{C}, \rho_\kappa)$ with $\kappa = 0$ and good rate function:

$$I(f) = \begin{cases} \frac{1}{2} \int_0^\infty \frac{(f'(t))^2}{\sigma^2(f(t))} dt, & \text{if } f(0) = x_0, \text{ } f \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. Existence of weak solution in (5.5) follows e.g. from Proposition 1 of [17]. Condition **I** holds by Theorem 1 in [17] and by extending f for $t \geq T$ by its value at T .

Consider the rate function

$$I_0^T(f^{(T)}) = \begin{cases} \frac{1}{2} \int_0^T \frac{(f'(t))^2}{\sigma^2(f(t))} dt, & \text{if } f(0) = x_0, \text{ } f \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

We verify condition **II**. Using (5.6) and applying Cauchy-Bunyakovskii inequality, we have (recall that $B_r^+ := \{f \in \mathbb{C} : \varlimsup_{T \rightarrow \infty} I_0^T(f^{(T)}) \leq r\}$)

$$\begin{aligned} \lim_{T \rightarrow \infty} \sup_{f \in B_r^+} \sup_{t \geq T} \frac{|f(t)|}{1+t} &= \lim_{T \rightarrow \infty} \sup_{f \in B_r^+} \sup_{t \geq T} \frac{1}{1+t} \left| \int_0^t f'(s) ds \right| \leq \lim_{T \rightarrow \infty} \sup_{f \in B_r^+} \sup_{t \geq T} \frac{t^{1/2}}{1+t} \left(\int_0^t (f'(s))^2 ds \right)^{1/2} \\ &\leq \lim_{T \rightarrow \infty} \sup_{f \in B_r^+} \frac{1}{T^{1/2}} \left(\lambda \int_0^T \frac{(f'(s))^2}{\sigma^2(f(s))} ds \right)^{1/2} \\ &= \lim_{T \rightarrow \infty} \sup_{f \in B_r^+} \frac{\sqrt{2\lambda}}{T^{1/2}} \sqrt{I_0^T(f^{(T)})} \leq \lim_{T \rightarrow \infty} \frac{\sqrt{2\lambda}r}{T^{1/2}} = 0. \end{aligned}$$

Condition **III** follows from (5.6) and Lemma 4.1. \square

5.2. Non-zero drift. Consider solution of SDE on half-line $[0, \infty)$

$$X_n(t) = x_0 + \int_0^t a(X_n(s)) ds + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s). \quad (5.7)$$

Lemma 5.2. *Suppose there exists $\lambda > 0$ such that for all $y \in R$*

$$|a(y)| + \sigma^2(y) \leq \lambda. \quad (5.8)$$

Then for any $\kappa > 0$, $N < \infty$ and $\varepsilon > 0$ there exists $T = T_{N,\varepsilon,\kappa} < \infty$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P} \left(\sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon \right) \leq -N. \quad (5.9)$$

Proof. Using condition (5.8) for $T > 1 \vee (\frac{4|x_0|}{\varepsilon} - 1)^{\frac{1}{1+\kappa}} \vee (\frac{4\lambda}{\varepsilon})^{\frac{1}{\kappa}}$ we obtain

$$\begin{aligned} & \mathbf{P} \left(\sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon \right) \\ & \leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{1 + t^{1+\kappa}} \left(|x_0| + \left| \int_0^t a(X_n(s)) ds \right| + \left| \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s) \right| \right) > \varepsilon \right) \\ & \leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{1 + t^{1+\kappa}} \left(|x_0| + \lambda t + \left| \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s) \right| \right) > \varepsilon \right) \\ & \leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{t^{1+\kappa}} \left| \int_0^t \sigma(X_n(s)) dw(s) \right| > \frac{\sqrt{n}\varepsilon}{2} \right) \leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{t} \left| \int_0^t \sigma(X_n(s)) dw(s) \right| > \frac{\sqrt{n}\varepsilon}{2} \right). \end{aligned}$$

The rest of the proof repeats that of Lemma 4.1. \square

Theorem 5.2. Let $a(y)$ and $\sigma(y)$ are functions of real argument such that for some $\lambda \geq 1$ and all $y, x \in R$

$$|a(y) - a(x)| + |\sigma(y) - \sigma(x)| \leq \lambda |y - x| \quad (5.10)$$

$$\frac{1}{\lambda} \leq \sigma^2(y) \leq \lambda, \quad |a(y)| \leq \lambda. \quad (5.11)$$

Then for any given $\kappa > 0$ the sequence X_n satisfies LDP in $(\mathbb{C}, \rho_\kappa)$ with good rate function:

$$I(f) = \begin{cases} \frac{1}{2} \int_0^\infty \frac{(f'(t) - a(f(t)))^2}{\sigma^2(f(t))} dt, & \text{if } f(0) = x_0, \text{ } f \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. Existence of a strong solution in (5.7) is assured by Theorem 1 in [14].

Condition **I** follows from [13] and that it is possible to extend f for $t \geq T$ by solution of differential equation:

$$g'(t) = a(g(t)), \quad g(T) = f(T), \quad t \geq T.$$

Condition **II** is verified similarly to that in Theorem 4.1. Condition **III** follows from (5.10), (5.11) and Lemma 4.2. \square

6. LARGE DEVIATIONS FOR CEV MODEL ON HALF-LINE

Consider $X_n(t)$, $t \geq 0$, that solves the following SDE (also known as the Constant Elasticity of Variance model, CEV).

$$X_n(t) = 1 + \int_0^t \mu X_n(s) ds + \frac{1}{n^{1-\gamma}} \int_0^t \sigma(X_n(s))^\gamma dw(s),$$

where μ and σ are arbitrary constants, $\gamma \in [1/2, 1)$, $n > 0$. Existence and uniqueness of strong solution is given e.g. in [15] and [8].

Lemma 6.1. *For any $N < \infty$ and $\varepsilon > 0$ there exists $T = T_{N,\varepsilon} < \infty$ such that*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2(1-\gamma)}} \ln \mathbf{P} \left(\sup_{t \geq T} \frac{X_n(t)}{e^{\mu t} (1+t)^{\frac{1}{1-\gamma}}} > \varepsilon \right) \leq -N. \quad (6.1)$$

Proof. Denote $\tau = \inf\{t \geq 0 : X_n(t) = 0\} \wedge \infty$. Using Itô's formula for $(X_n(t)e^{-\mu t})^{1-\gamma}$, $t \in [0, \tau)$, we have

$$\begin{aligned} (X_n(t)e^{-\mu t})^{1-\gamma} &= 1 + \frac{1}{n^{1-\gamma}} \int_0^t \sigma(1-\gamma) e^{-\mu(1-\gamma)s} dw(s) \\ &\quad - \frac{1}{2n^{2(1-\gamma)}} \int_0^t \gamma(1-\gamma) \frac{\sigma^2 e^{-\mu(1-\gamma)s}}{(X_n(s))^{1-\gamma}} ds, \quad t \in [0, \tau). \end{aligned}$$

Since $X_n(t)$ is non-negative with probability 1

$$\begin{aligned} (X_n(t)e^{-\mu t})^{1-\gamma} &\leq 1 + \frac{1}{n^{1-\gamma}} \int_0^t \frac{\sigma(1-\gamma)}{e^{\mu(1-\gamma)s}} dw(s) \\ &\leq 1 + \frac{1}{n^{1-\gamma}} \left| \int_0^t \frac{\sigma(1-\gamma)}{e^{\mu(1-\gamma)s}} dw(s) \right| \text{ a.s., } t \in [0, \tau). \end{aligned}$$

Since $X_n(t) \equiv 0$ for $t \geq \tau$, the above inequality trivially holds for $t \in [0, \infty)$.

Therefore for $T > 2/\varepsilon^{1-\gamma}$

$$\begin{aligned} \mathbf{P} \left(\sup_{t \geq T} \frac{X_n(t)}{e^{\mu t} (1+t)^{\frac{1}{1-\gamma}}} > \varepsilon \right) &= \mathbf{P} \left(\sup_{t \geq T} \frac{(X_n(t)e^{-\mu t})^{1-\gamma}}{1+t} > \varepsilon^{1-\gamma} \right) \\ &\leq \mathbf{P} \left(\sup_{t \geq T} \left(\frac{1}{1+t} + \frac{1}{n^{1-\gamma}(1+t)} \left| \int_0^t \frac{\sigma(1-\gamma)}{e^{\mu(1-\gamma)s}} dw(s) \right| \right) > \varepsilon^{1-\gamma} \right) \\ &\leq \mathbf{P} \left(\sup_{t \geq T} \frac{1}{n^{1-\gamma}(1+t)} \left| \int_0^t \frac{\sigma(1-\gamma)}{e^{\mu(1-\gamma)s}} dw(s) \right| > \frac{\varepsilon^{1-\gamma}}{2} \right). \end{aligned}$$

Since $|\sigma(1-\gamma)e^{-\mu(1-\gamma)s}| \leq \sigma(1-\gamma)$, using Lemma 4.1 we obtain (6.1). \square

Denote by \mathbb{C}^+ the set of functions in $f \in \mathbb{C}$, such that $f(0) = 1$, $f(t) \geq 0$ for all $t \geq 0$. Define a metric in \mathbb{C}^+ by

$$\rho^{\mu,\gamma}(f, g) := \sup_{t \geq 0} \frac{|f(t) - g(t)|}{e^{\mu t} (1+t)^{\frac{1}{1-\gamma}}}.$$

It is obvious that the space $(\mathbb{C}^+, \rho^{\mu,\gamma})$ is Polish.

Theorem 6.1. *The process X_n satisfies LDP in space $(\mathbb{C}^+, \rho^{\mu,\gamma})$ with rate $\frac{1}{n^{2(1-\gamma)}}$ and good rate function*

$$I(f) = \begin{cases} \frac{1}{2} \int_0^{\Theta(f)} \frac{(f'(t) - \mu f(t))^2}{(f(t))^{2\gamma}} dt, & \text{if } f(0) = 1, f \text{ is absolutely continuous,} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\Theta(f) = \inf\{t : f(t) = 0\}$.

Proof. Condition **I** follows from [15] and that one can extend f for $t \geq T$ by the solution of the differential equation

$$g'(t) = \mu g(t), \quad g(T) = f(T), \quad t \geq T.$$

We verify condition **II**. Let

$$\overline{\lim}_{T \rightarrow \infty} I^T(f^{(T)}) = \frac{r^2}{2} < \infty.$$

Write $f(t)$ as $f(t) = e^{\mu t} g(t)$, $g(0) = 1$. Then

$$I^T(f^{(T)}) = \frac{1}{2} \int_0^T e^{2\mu(1-\gamma)t} \left(\frac{g'(t)}{g^\gamma(t)} \right)^2 dt.$$

Denote

$$u^2(t) = e^{2\mu(1-\gamma)t} \left(\frac{g'(t)}{g^\gamma(t)} \right)^2, \quad \overline{\lim}_{T \rightarrow \infty} \int_0^T u^2(t) dt = r^2. \quad (6.2)$$

Then

$$\frac{g'(t)}{g^\gamma(t)} = e^{-\mu(1-\gamma)t} u(t), \quad g(0) = 1.$$

Solving, we have

$$g^{1-\gamma}(t) = 1 + (1-\gamma) \int_0^t e^{-\mu(1-\gamma)s} u(s) ds.$$

Using Cauchy-Bunyakovskii inequality and (6.2) we have

$$|g^{1-\gamma}(t)| \leq 1 + (1-\gamma) \left(\int_0^t e^{-2\mu(1-\gamma)s} ds \int_0^t u^2(s) ds \right)^{1/2} \leq 1 + r \left(\frac{1-\gamma}{2\mu} \right)^{1/2}.$$

Hence

$$|f(t)| \leq e^{\mu t} \left(1 + r \left(\frac{1-\gamma}{2\mu} \right)^{1/2} \right)^{\frac{1}{1-\gamma}}.$$

It now follows that

$$\overline{\lim}_{T \rightarrow \infty} \sup_{T \geq t} \frac{|f(t)|}{e^{\mu t} (1+t)^{\frac{1}{1-\gamma}}} = 0.$$

Condition **III** follows from Lemma 5.1. Theorem 5.1 is now proved. \square

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REFERENCES

- [1] Borovkov A.A. *Probability Theory*. Moscow, Editorial URSS, 2009, 470 p. (in Russian)
- [2] Borovkov A.A. *Asymptotic Analysis of Random Walks. Quickly Decreasing Jumps*. Fizmatlit, Moscow, 2013. (in Russian)
- [3] Borovkov A.A. (1967) Boundary problems for random walks and large deviations in function spaces. *Theor. Probability Appl.* Vol. 12, No. 4, p. 575-595.
- [4] Borovkov A.A. and Mogulskii A.A. (2010) On large deviation principles in metric spaces. *Sibirsk. Mat. Zh.* Vol. 51, p. 1251-1269.
- [5] Borovkov A.A. Mogulskii A.A. *Large deviations and testing statistical hypothesis*. Nauka, Novosibirsk, 1992, 222 p. (in Russian)

- [6] Borovkov A.A. and Mogulskii A.A. (2013) Moderately large deviation principles for trajectories of random walks and processes with independent increments. *Theory Probab. Appl.* Vol. 58, No. 4, p. 648–671. (in Russian)
- [7] Budkov D.S. and Makhno S.Ya. (2007) Functional iterated logarithm law for a Wiener process. *Theory of Stochastic Processes*. Vol. 13(29), No. 3, p. 22-28.
- [8] F. Delbaen and H. Shirakawa. (2002) A note on option pricing for the constant elasticity of variance model. *Asia-Pacific Financial Markets*, No. 9(2), p.85–99.
- [9] Dembo A. Zeitouni O. *Large Deviations Techniques and Applications*. Springer, 2nd edition, 1998.
- [10] Deuschel J.D. and Stroock D.W. *Large Deviations*. Academic Press, Boston 1989.
- [11] Dobrushin R.L. and Pecherskij E.A. (1998) Large deviations for random processes with independent increments on infinite intervals. *Probl. Inf. Transm.* 34, No.4, p. 354-382.
- [12] Feng J. and Kurtz T. *Large deviations for stochastic processes*. American Mathematical Society, 2006.
- [13] Freidlin M. and Wentzell A. *Random Perturbations of Dynamical Systems*. Springer-Verlag, New York, 1998.
- [14] Gihman I.I. and Skorokhod A.V. *Stochastic Differential Equations*. Kiev, Naukova dumka, 1968, 355 p. (in Russian)
- [15] Klebaner F. and Liptser R. (2011) Asymptotic Analysis of Ruin in the Constant Elasticity of Variance Model. *Theory Probab. Appl.* Vol. 55, No 2, p. 291-297.
- [16] Kolmogorov A.N. and Fomin S.V. *Elements of the theory of functions and functional analysis*. Nauka, Moscow, 1976, 543 p. (in Russian)
- [17] Kulik A.M. and Soboleva D.D. Large deviations for one-dimensional SDE with discontinuous diffusion coefficient. *Theory of Stochastic Processes* Vol. 18(34), No. 1, 2012, p. 101-110.
- [18] Mogulskii A.A. (1976) Large Deviations for Trajectories of Multi-Dimensional Random Walks. *Theory Probab. Appl.* Vol. 21, No. 2, p. 300-315.
- [19] Puhalskii A.A. *Large deviations and idempotent probability*. Chapman and Hall/ CRC Monographs and Surveys in Pure and Applied Mathematics, 119. Chapman and Hall/ CRC, Boca Raton, FL, 2001.
- [20] Puhalskii A.A. *Large deviations for stochastic processes. LMS/EPSRC Short Course: Stochastic Stability, Large Deviations and Coupling Methods*. Heriot-Watt University, Edinburgh, 4-6 September 2006.

SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, CLAYTON, VIC 3800, AUSTRALIA.

E-mail address: `fima.klebaner@monash.edu`

NOVOSIBIRSK STATE UNIVERSITY, NOVOSIBIRSK, 2 PIROGOVA STR., 630090, RUSSIA.

E-mail address: `omboldovskaya@mail.ru`

SOBOLEV INSTITUTE OF MATHEMATICS OF THE SIBERIAN BRANCH OF THE RAS, NOVOSIBIRSK, 4 ACAD. KOPTYUG AVENUE, 630090, RUSSIA.

E-mail address: `mogul@math.nsc.ru`